δ-SUPERDERIVATIONS OF SIMPLE FINITE-DIMENSIONAL JORDAN AND LIE SUPERALGEBRAS

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Abstract:

We introduce the concept of a δ-superderivation of a superalgebra. δ-Derivations of Cartan-type Lie superalgebras are treated, as well as δ-superderivations of simple finite-dimensional Lie superalgebras and Jordan superalgebras over an algebraically closed field of characteristic 0. We give a complete description of $\frac{1}{2}$-derivations for Cartan-type Lie superalgebras. It is proved that nontrivial δ-(super)derivations are missing on the given classes of superalgebras, and as a consequence, δ-superderivations are shown to be trivial on simple finite-dimensional noncommutative Jordan superalgebras of degree at least 2 over an algebraically closed field of characteristic 0. Also we consider δ-derivations of unital flexible and semisimple finite-dimensional Jordan algebras over a field of characteristic not 2.

INTRODUCTION

The notion of a derivation of an algebra was generalized by many mathematicians in a number of different directions. Thus, in [1], we can find the definition of a δ-derivation of an algebra. Recall that with $\delta \in F$ fixed, a *δ-derivation* of an algebra A is conceived of as a linear map ϕ satisfying the condition $\phi(xy) = \delta(\phi(x)y + x\phi(y))$ for arbitrary elements $x, y \in A$. In [1], also, $\frac{1}{2}$-derivations are described for an arbitrary primary Lie Φ-algebra A ($\frac{1}{6} \in \Phi$) with a nondegenerate symmetric invariant bilinear form. Namely, it was proved that the linear map $\phi : A \to A$ is a $\frac{1}{2}$-derivation iff $\phi \in \Gamma(A)$, where $\Gamma(A)$ is the centroid of an algebra A. This implies that if A is a central simple Lie algebra having a nondegenerate symmetric invariant bilinear form over a field of characteristic $p \neq 2, 3$, then any $\frac{1}{2}$-derivation ϕ is represented as $\phi(x) = \alpha x$, $\alpha \in \Phi$. In [2], it was proved that every primary Lie Φ-algebra does not have a nonzero δ-derivation if $\delta \neq -1, 0, \frac{1}{2}, 1$, and that every primary Lie Φ-algebra A ($\frac{1}{6} \in \Phi$) with a nonzero antiderivation satisfies the identity $[(yz)(tx)]x + [(yx)(zx)]t = 0$ and is a three-dimensional central simple algebra over a field of quotients of the center $Z_R(A)$ of its right multiplication algebra $R(A)$. In [2], too, we can find an example of a nontrivial $\frac{1}{2}$-derivation for a Witt algebra W_1, by which is meant a $\frac{1}{2}$-derivation which is not an element of the centroid of W_1. δ-Derivations of primary alternative and non-Lie Mal'tsev Φ-algebras with restrictions on an operator ring Φ were described in [3]. It turns out that algebras in these classes have no nonzero δ-derivation if $\delta \neq 0, \frac{1}{2}, 1$.

In [4], δ-derivations of semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic other than 2 were characterized, as well as simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0. δ-Derivations of classical Lie superalgebras were described in [5]. Namely, nonzero δ-derivations were shown

not to exist for $\delta \neq 0, \frac{1}{2}, 1$ and $\frac{1}{2}$-derivations of the given algebras and superalgebras were described out. Also δ-derivations (and δ-superderivations) of prime Lie superalgebras considered P. Zusmanovich [15]. He was proved that every primary Lie superalgebra does not have a nonzero δ-derivation if $\delta \neq -1, 0, \frac{1}{2}, 1$.

In the present paper, the concept of a δ-superderivation of a superalgebra is introduced. We consider δ-derivations of Cartan-type Lie superalgebras, and also δ-superderivations of simple finite-dimensional Lie superalgebras and of Jordan superalgebras over an algebraically closed field of characteristic 0. A complete description of $\frac{1}{2}$-derivations is furnished for Cartan-type Lie superalgebras. We prove that nontrivial δ-(super)derivations are missing on the given classes of superalgebras, and as a consequence, infer that δ-superderivations are trivial on simple finite-dimensional noncommutative Jordan superalgebras of degree at least 2 over an algebraically closed field of characteristic 0. Also we consider δ-derivations for unital flexible and semisimple finite-dimensional Jordan algebras over a field of characteristic not 2.

1. BASIC FACTS AND DEFINITIONS

Let Γ be the Grassmann algebra over F generated by elements $1, e_1, \ldots, e_n, \ldots$ and defined by relations $e_i^2 = 0$ and $e_i e_j = -e_j e_i$. Products $1, e_{i_1} e_{i_2} \ldots e_{i_k}$, $i_1 < i_2 < \ldots < i_k$, form a basis for the algebra Γ over F. Denote by $\Gamma_{\overline{0}}$ and $\Gamma_{\overline{1}}$ subspaces generated by products of, respectively, even and odd lengths. Then Γ is representable as a direct sum of those subspaces (written $\Gamma = \Gamma_{\overline{0}} \oplus \Gamma_{\overline{1}}$), and the following relations hold: $\Gamma_{\overline{i}} \Gamma_{\overline{j}} \subseteq \Gamma_{\overline{i}+\overline{j}\,(\mathrm{mod} 2)}$, $i, j = 0, 1$. In other words, Γ is a $\mathbb{Z}_2$-graded algebra (or superalgebra) over F.

Now let $A = A_{\overline{0}} \oplus A_{\overline{1}}$ be an arbitrary superalgebra over F. Consider the tensor product $\Gamma \otimes A$ of F-algebras. Its subalgebra

$$\Gamma(A) = \Gamma_{\overline{0}} \otimes A_{\overline{0}} + \Gamma_{\overline{1}} \otimes A_{\overline{1}}$$

is called the *Grassmann envelope* of a superalgebra A.

Let Ω be some variety of algebras over F. A superalgebra $A = A_{\overline{0}} \oplus A_{\overline{1}}$ is called a *Ω-superalgebra* if its Grassmann envelope $\Gamma(A)$ is an algebra in Ω. *Classical Lie superalgebras* are simple finite-dimensional Lie superalgebras $G = G_{\overline{0}} \oplus G_{\overline{1}}$ over an algebraically closed field of characteristic 0, where $G_{\overline{1}}$ is a completely reducible $G_{\overline{0}}$-module. Simple finite-dimensional Lie superalgebras over an algebraically closed field of characteristic 0 that are not classical are called *Cartan-type Lie superalgebras*. It follows from [6] that Cartan superalgebras are exhausted by superalgebras of the forms $W(n)$, $n \geqslant 2$, $S(n)$, $n \geqslant 3$, $\widetilde{S}(2n)$, $n \geqslant 2$, and $H(n)$, $n \geqslant 5$.

We start by defining $W(n)$. Let $\Lambda(n)$ be a Grassmann superalgebra with a set of generators $\xi_1, \ldots, \xi_n$. We define $\mathrm{der}\Lambda(n)$ as $W(n)$. Every derivation $D \in W(n)$ is representable as

$$D = \sum_i P_i \frac{\partial}{\partial \xi_i}, \quad P_i \in \Lambda(n),$$

where $\frac{\partial}{\partial \xi_i}$ is a derivation given by the rule $\frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}$. Let $\deg \xi_i = 1$. Then $W(n) = \bigoplus\limits_{k \geqslant -1} W(n)_k$, where

$$W(n)_k = \left\{ \sum P_i \frac{\partial}{\partial \xi_i} \;\middle|\; \deg P_i = k+1,\ i = 1, \ldots, n \right\}.$$

The superalgebras $S(n)$, $\widetilde{S}(n)$, $\widetilde{H}(n)$, and $H(n)$ are subsuperalgebras of $W(n)$ defined as follows:

$$\begin{aligned}
S(n) &= \left\{ \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_i} \;\middle|\; f \in \Lambda(n),\, i,j = 1,\ldots,n \right\};\\
\widetilde{S}(n) &= \left\{ (1-\xi_1\ldots\xi_n)\left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_i}\right) \;\middle|\; f \in \Lambda(n),\, i,j = 1,\ldots,n \right\},\\
&\quad n = 2l;\\
\widetilde{H}(n) &= \left\{ D_f = \sum_i \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_i} \;\middle|\; f \in \Lambda(n),\, f(0)=0,\, i = 1,\ldots,n \right\};\\
H(n) &= [\widetilde{H}(n), \widetilde{H}(n)].
\end{aligned}$$

Root systems for the given superalgebras are described in [7].

For the case $\Theta = W(n)$, elements $h_i = \xi_i\frac{\partial}{\partial\xi_i}$ form a basis for a Cartan subalgebra, and $\Theta_0 = gl_n$. Elements ε_i constitute a basis dual to h_i. A root system shows up as follows:

$$\Delta = \Big\{ \varepsilon_{i_1} + \ldots + \varepsilon_{i_k}, \varepsilon_{i_1} + \ldots + \varepsilon_{i_k} - \varepsilon_j \Big| \quad i_r \neq i_s,\, j \neq i_r,\, 0 \leqslant k \leqslant n-1,\, 1 \leqslant j \leqslant n \Big\}.$$

The superalgebras $\Theta = S(n), \widetilde{S}(n)$ ($n = 2l$ in the latter case) have a Cartan subalgebra H generated by elements $h_{ij} = \xi_i\frac{\partial}{\partial\xi_i} - \xi_j\frac{\partial}{\partial\xi_j}$. It is known that $\Theta_0 = sl_n$. Elements ε_i form a basis dual to h_i, and $\varepsilon_1 + \ldots + \varepsilon_n = 0$. A root system is the following:

$$\Delta = \Big\{ \varepsilon_{i_1} + \ldots + \varepsilon_{i_k}, \varepsilon_{i_1} + \ldots + \varepsilon_{i_l} - \varepsilon_j \;\Big|\; i_r \neq i_s,\, j \neq i_s,\quad 1 \leqslant k \leqslant n-2,\, 0 \leqslant l \leqslant n-1,\, 1 \leqslant j \leqslant n \Big\}.$$

If $\Theta = H(n)$ then $\Theta_0 = D_l$, for $n = 2l$, and $\Theta_0 = B_l$ for $n = 2l+1$. The superalgebra Θ_0 has a Cartan subalgebra H generated by elements $D_{\xi_i\xi_{i+l}}$. Let $h_i = \sqrt{-1}D_{\xi_i\xi_{i+l}}$ and ε_i be elements of the basis dual to h_i. A root system is described thus:

$$\Delta = \left\{ \varepsilon_{i_1} + \ldots + \varepsilon_{i_t} - \varepsilon_{j_1} - \ldots - \varepsilon_{j_s} \;\middle|\; i_r \neq i_p,\, j_r \neq j_p,\, i_r \neq j_p,\, 0 \leqslant t,s \leqslant l \right\}.$$

Note that if $\Theta = S(n), \widetilde{S}(n), H(n)$ then Θ_k is defined similarly to how is $W(n)_k$ defined for $W(n)$.

2. δ-DERIVATIONS OF CARTAN-TYPE LIE SUPERALGEBRAS

Let $\delta \in F$. A linear mapping ϕ of a superalgebra A is called a δ-*derivation* if, for arbitrary elements $x, y \in A$,

$$\phi(xy) = \delta(x\phi(y) + \phi(x)y).$$

A definition for a 1-derivation coincides with the usual definition of a derivation. A 0-derivation is any endomorphism ϕ of A such that $\phi(A^2) = 0$. A *nontrivial* δ-*derivation* is a nonzero δ-derivation ϕ which is not a 1- or 0-derivation, nor a $\frac{1}{2}$-derivation such that $\phi \in \Gamma(A)$. We are interested in how nontrivial δ-derivations act on Cartan-type Lie superalgebras.

By G_β we mean a root subspace corresponding to a root β, and g_β is conceived of as an element of that subspace.

LEMMA 1. Let $G = G_{\overline{0}} \oplus G_{\overline{1}}$ be a Cartan-type Lie superalgebra and ϕ a nontrivial δ-derivation of the superalgebra G. Then $\phi(G_{\overline{i}}) \subseteq G_{\overline{i}}$.

Proof. In [6], it was shown that $G_{\overline{0}} = [G_{\overline{1}}, G_{\overline{1}}]$. Consequently, for any $x \in G_{\overline{0}}$, we have

$$x = \sum_{i=1}^{n_x} \left(\left(\frac{v_i + z_i}{2} \right)^2 - \left(\frac{v_i - z_i}{2} \right)^2 \right),$$

where $z_i, v_i \in G_{\overline{1}}$. Note that for $y \in G_{\overline{1}}$ and $\phi(y) = y_0 + y_1$, $y_i \in G_{\overline{i}}$,

$$\phi(y^2) = \delta((y_0 + y_1)y + y(y_0 + y_1)) = 2\delta y_1 y \in G_{\overline{0}}.$$

Hence $\phi(G_{\overline{0}}) \subseteq G_{\overline{0}}$.

Let H be a Cartan subalgebra, $\beta \neq 0$, $g_\beta \in G_{\overline{1}} \cap G_\beta$, $h \in H$, and $\phi(g_\beta) = \sum_{\gamma \in \Delta} g_\gamma + h_\beta$, where $h_\beta \in H$. We obtain the following chain of equalities:

$$\sum_{\gamma \in \Delta} \beta(h) g_\gamma + \beta(h) h_\beta = \phi(\beta(h) g_\beta) = \phi(h g_\beta)$$

$$= \delta(\phi(h) g_\beta + h\phi(g_\beta)) = \delta \left(\phi(h) g_\beta + \sum_{\gamma \in \Delta} \gamma(h) g_\gamma \right).$$

Since $\phi(h) \in G_{\overline{0}}$, $\phi(h) g_\beta \in G_{\overline{1}}$, and $h \in H$ is arbitrary, we have $\beta = \delta\gamma$ for $\gamma \in \Delta_{\overline{0}}$. This, in view of the form of root systems for Cartan-type Lie superalgebras, yields $\phi(G_{\overline{1}}) \subseteq G_{\overline{1}}$. The lemma is proved.

Let $\Theta = \bigoplus_{i=-1}^{n} \Theta_i$ be a Cartan-type Lie superalgebra. We will treat $\psi : \Theta_0 \to \Theta_0$ as a restriction of a δ-derivation ϕ to Θ_0. In view of Lemma 1, it is clear that for $x \in \Theta_0$, $\phi(x) = \psi(x) + \sum_{i=1}^{[n/2]} b_{2i}^x$, where $b_{2i}^x \in \Theta_{2i}$. It is easy to see that ψ is a δ-derivation of the algebra Θ_0. Relying on results in [1] and the structure of algebras Θ_0, we conclude that for $\Theta = S(n), \widetilde{S}(n), H(n)$, if $\delta = \frac{1}{2}$ then $\psi(g_0) = \sigma g_0$, $g_0 \in \Theta_0$, $\sigma \in F$, and if $\delta \neq \frac{1}{2}$ then $\psi(\Theta_0) = 0$. It follows from [5, Lemma 7] that for the case where $\Theta = W(n)$, it is true that $\psi(a) = \sigma a$, $\sigma \in F$, with $a = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}$.

LEMMA 2. The superalgebra $W(n)$ has no nontrivial δ-derivations.

Proof. It is easy to see that in the above terms, for $x_0, y_0 \in \Theta_0$, we have

$$0 = \phi(a x_0) = \delta \left(\sum_{i=1}^{[n/2]} b_{2i}^a x_0 + a \sum_{i=1}^{[n/2]} b_{2i}^{x_0} \right),$$

which entails $b_{2i}^{x_0} = -\frac{1}{2i} b_{2i}^a x_0$. This implies

$$\sum_{i=1}^{[n/2]} \frac{1}{2i} b_{2i}^a (x_0 y_0) = \psi(x_0 y_0) - \phi(x_0 y_0) = \delta \sum_{i=1}^{[n/2]} \frac{1}{2i} ((b_{2i}^a x_0) y_0 + x_0 (b_{2i}^a y_0)) = \delta \sum_{i=1}^{[n/2]} \frac{1}{2i} b_{2i}^a (x_0 y_0),$$

so $b^a_{2i}=0$, and hence $\phi(a)=\sigma a$ and $\phi(x_0)=\psi(x_0)$.

Note that if $\phi(g_{\varepsilon_i})=\sum\limits_{j=0}^{[n/2]} b^{g_{\varepsilon_i}}_{2j-1}$, where $g_{\varepsilon_i}\in\Theta_1$, then

$$\phi(g_{\varepsilon_i})=\delta(\sigma a g_{\varepsilon_i}+a\phi(g_{\varepsilon_i}))=\delta\left(\sigma g_{\varepsilon_i}+\sum_{j=0}^{[n/2]}(2j-1)b^{g_{\varepsilon_i}}_{2j-1}\right),$$

so $b^{g_{\varepsilon_i}}_{2j-1}=\delta(2j-1)b^{g_{\varepsilon_i}}_{2j-1}$, where $j\neq 1$, and $b^{g_{\varepsilon_i}}_1=\delta(\sigma g_{\varepsilon_i}+b^{g_{\varepsilon_i}}_1)$, whence $\phi(g_{\varepsilon_i})=\frac{\sigma\delta}{1-\delta}g_{\varepsilon_i}+b^{g_{\varepsilon_i}}_{\frac{1}{\delta}}$ if $\frac{1}{\delta}=2j-1$ and $j\in\{0,\dots,[n/2]\}$.

It is worth observing that $\phi(g_{\varepsilon_i})=\phi(h_{ij}g_{\varepsilon_i})=\delta(\psi(h_{ij})g_{\varepsilon_i}+h_{ij}(\frac{\sigma\delta}{1-\delta}g_{\varepsilon_i}+b^{g_{\varepsilon_i}}_{\frac{1}{\delta}}))$; i.e., $b^{g_{\varepsilon_i}}_{\frac{1}{\delta}}=\delta h_{ij}b^{g_{\varepsilon_i}}_{\frac{1}{\delta}}$. Keeping in mind that $b^{g_{\varepsilon_i}}_{\frac{1}{\delta}}\in W(n)_{2l-1}$ and $h_{ij}g_\alpha=0,\pm g_\alpha,\pm 2g_\alpha$, we arrive at $\delta=-1$ and $\phi(g_{\varepsilon_i})=\frac{\sigma\delta}{1-\delta}g_{\varepsilon_i}+\beta^{g_{\varepsilon_i}}_{g_{-\varepsilon_i}}g_{-\varepsilon_i}$.

For $\delta=-1$, the fact that $0=\phi(g^2_{\varepsilon_i})=-2\phi(g_{\varepsilon_i})g_{\varepsilon_i}=\beta^{g_{\varepsilon_i}}_{-\varepsilon_i}g_{-\varepsilon_i}g_{\varepsilon_i}$ implies $\phi(g_{\varepsilon_i})=\frac{\sigma\delta}{1-\delta}g_{\varepsilon_i}$. Similarly, $\phi(g_{-\varepsilon_i})=\frac{\sigma\delta}{1-\delta}g_{-\varepsilon_i}$. This entails

$$\frac{\sigma\delta}{1-\delta}g_{\varepsilon_i}=\phi(g_{\varepsilon_i})=\phi((g_{\varepsilon_i}g_{-\varepsilon_j})g_{\varepsilon_j})=\delta(\phi(g_{\varepsilon_i}g_{-\varepsilon_j})g_{\varepsilon_j}+(g_{\varepsilon_i}g_{-\varepsilon_j})\phi(g_{\varepsilon_j}))=(2\delta^2+\delta)\frac{\sigma\delta}{1-\delta}g_{\varepsilon_i}.$$

Clearly, $\sigma\neq 0$ only if $\delta=-1,\frac{1}{2}$; otherwise $\sigma=0$.

If $\delta=-1$ then $\phi(g_{\varepsilon_i-\varepsilon_j})=\phi(g_{\varepsilon_i}g_{-\varepsilon_j})=\sigma g_{\varepsilon_i-\varepsilon_j}$ and $\phi(g_{\pm\varepsilon_i})=-\frac{\sigma}{2}g_{\pm\varepsilon_i}$. It remains to note that

$$\phi(h_{ij})=\phi\left(\sum x^0_k y^0_k\right)=-2\sigma h_{ij}\quad\text{and}\quad\phi(h_{ij})=\phi\left(\sum x^{-1}_l y^1_l\right)=\sigma h_{ij},$$

where $x^k_l,y^k_l\in W(n)_k$, whence $\sigma=0$.

If $\delta\neq\frac{1}{2}$, then $\phi(g_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_{k-1}}\pm\varepsilon_{i_k}})=0$, since $\phi(g_{\pm\varepsilon_i})=0$. But if $\delta=\frac{1}{2}$, then $\phi(g_{\pm\varepsilon_i})=\sigma g_{\pm\varepsilon_i}$. Hence $\phi(g_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_{k-1}}\pm\varepsilon_{i_k}})=\sigma g_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_{k-1}}\pm\varepsilon_{i_k}}$. The argument above implies that ϕ is trivial. The lemma is proved.

Below, for a root $\alpha=\sum\limits_{i\in I_0}\varepsilon_i-\sum\limits_{j\in I_1}\varepsilon_j$, $I_0\cap I_1=\varnothing$, by $\alpha\supseteq(-1)^k\varepsilon_t$ and $\alpha\not\supseteq(-1)^k\varepsilon_t$ we mean $t\in I_k$ and $t\notin I_k$, respectively.

LEMMA 3. The superalgebras $S(n)$ and $\widetilde{S}(n)$ have no nontrivial δ-derivations.

Proof. In what follows, we denote both superalgebras $S(n)$ and $\widetilde{S}(n)$ by $S(n)$, and α^* will signify the fact that $\alpha^*\not\supseteq\pm\varepsilon_i,\pm\varepsilon_j$. Using the argument above, we conclude that

$$\phi(h_{ij})=\psi(h_{ij})+\sum_{\alpha\in\Delta_0}b^{h_{ij}}_\alpha,\,b^{h_{ij}}_\alpha\in\bigcup_{k=1}^{[n/2]}S(n)_{2k}.$$

It is easy to see that

$$0=\phi(h_{ij}h_{ik})=\delta(\phi(h_{ij})h_{ik}+h_{ij}\phi(h_{ik})),$$

whence $b^{h_{ij}}_{\alpha^*}=b^{h_{ij}}_{\varepsilon_i+\varepsilon_j+\alpha^*}=0$, and $b^{h_{ik}}_{\varepsilon_i+\alpha^*}=0$ and $2b^{h_{ij}}_{-\varepsilon_i+\alpha^*}=b^{h_{ik}}_{-\varepsilon_i+\alpha^*}$ for $\alpha^*\supseteq\varepsilon_k$.

Let α^* be such that $\alpha^*\supseteq\varepsilon_k$. Then $\phi(g_{\varepsilon_i})=\delta(\phi(h_{ik})g_{\varepsilon_i}+h_{ik}\phi(g_{\varepsilon_i}))$. Hence

$$b^{g_{\varepsilon_i}}_{\alpha^*}=\delta(b^{h_{ik}}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i}-b^{g_{\varepsilon_i}}_{\alpha^*}).$$

The fact that $\phi(g_{\varepsilon_i})=\delta(\phi(h_{ij})g_{\varepsilon_i}+h_{ij}\phi(g_{\varepsilon_i}))$ implies

$$b^{g_{\varepsilon_i}}_{\alpha^*}=\delta b^{h_{ij}}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i}.$$

The relations obtained yield $(1+\delta)b^{h_{ij}}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i} = b^{h_{ik}}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i} = 2b^{h_{ij}}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i}$; i.e., $b^{h_{ij}}_{-\varepsilon_i+\alpha^*} = 0$.

Obviously, $\phi(g_{-\varepsilon_i}) = -\delta(\phi(h_{ij})g_{-\varepsilon_i} + h_{ij}\phi(g_{-\varepsilon_i}))$, and so

$$b^{g_{-\varepsilon_i}}_{\alpha^*} = -\delta b^{h_{ij}}_{\varepsilon_i+\alpha^*}g_{-\varepsilon_i}.$$

The equality $\phi(g_{-\varepsilon_i}) = -\delta(\phi(h_{ik})g_{-\varepsilon_i} + h_{ik}\phi(g_{-\varepsilon_i}))$ gives

$$b^{g_{-\varepsilon_i}}_{\alpha^*} = \delta(b^{h_{ik}}_{\varepsilon_i+\alpha^*}g_{-\varepsilon_i} - b^{g_{-\varepsilon_i}}_{\alpha^*}),$$

whence $(1-\delta)b^{h_{ij}}_{\varepsilon_i+\alpha^*} = b^{h_{ik}}_{\varepsilon_i+\alpha^*} = b^{h_{ik}}_{\varepsilon_i+\varepsilon_k+\gamma^*} = 0$, where $\gamma^* \not\supseteq \pm\varepsilon_i, \pm\varepsilon_k$; i.e., $b^{h_{ij}}_{\varepsilon_i+\alpha^*} = 0$.

We claim that $b^{h_{ij}}_{\varepsilon_i-\varepsilon_j+\alpha^*} = 0$. To prove this, consider

$$2\phi(g_{\varepsilon_j-\varepsilon_i}) = -\delta(\phi(h_{ij})g_{\varepsilon_j-\varepsilon_i} + h_{ij}\phi(g_{\varepsilon_j-\varepsilon_i})).$$

This yields

$$2b^{g_{\varepsilon_j-\varepsilon_i}}_{\alpha^*} = -2\delta b^{h_{ij}}_{\varepsilon_i-\varepsilon_j+\alpha^*}g_{\varepsilon_j-\varepsilon_i}.$$

On the other hand, $\phi(g_{\varepsilon_j-\varepsilon_i}) = -\delta(\phi(h_{ik})g_{\varepsilon_j-\varepsilon_i} + h_{ik}\phi(g_{\varepsilon_j-\varepsilon_i}))$ gives

$$b^{g_{\varepsilon_j-\varepsilon_i}}_{\alpha^*} = -\delta(b^{h_{ik}}_{\varepsilon_i-\varepsilon_j+\alpha^*}g_{\varepsilon_j-\varepsilon_i} - b^{g_{\varepsilon_j-\varepsilon_i}}_{\alpha^*}).$$

Since $\alpha^* \supseteq \varepsilon_k$, we define $\gamma^* = \alpha^* - \varepsilon_k - \varepsilon_j$, where $\gamma^* \not\supseteq \pm\varepsilon_i, \pm\varepsilon_k$; hence $b^{h_{ik}}_{\varepsilon_i+\varepsilon_k+\gamma^*} = 0$ and $b^{h_{ik}}_{\varepsilon_i-\varepsilon_j+\alpha^*} = 0$. Consequently, $b^{g_{\varepsilon_j-\varepsilon_i}}_{\alpha^*} = 0$ and $b^{h_{ij}}_{\varepsilon_i-\varepsilon_j+\alpha^*} = 0$.

The equalities $b^{h_{ij}}_{\varepsilon_j-\varepsilon_i+\alpha^*} = b^{h_{ij}}_{\pm\varepsilon_j+\alpha^*} = 0$ are obtained by substituting $i \leftrightarrow j$ and $h_{ij} = -h_{ji}$ simultaneously. Thus $\phi(h_{ij}) = \psi(h_{ij})$.

If $\delta \neq \frac{1}{2}$, then the fact that $\phi(g_{\pm\varepsilon_i}) = \pm\delta h_{ik}\phi(g_{\pm\varepsilon_i})$ implies $\phi(g_{\pm\varepsilon_i}) = b^{g_{\pm\varepsilon_i}}_{\mp\varepsilon_i}$ and $\delta = -1$, since k is arbitrary. Note that $0 = \phi(g^2_{\pm\varepsilon_i}) = 2\delta\phi(g_{\pm\varepsilon_i})g_{\pm\varepsilon_i} = 2\delta b^{g_{\pm\varepsilon_i}}_{\mp\varepsilon_i}g_{\pm\varepsilon_i}$, whence $b^{g_{\pm\varepsilon_i}}_{\mp\varepsilon_i} = 0$. Taking into account that $g_{\pm\varepsilon_1}, \ldots, g_{\pm\varepsilon_n}$ are generators for $S(n)$, we have $\phi(S(n)) = 0$.

If $\delta = \frac{1}{2}$, then $\phi(g_{\pm\varepsilon_i}) = \frac{1}{2}(\sigma g_{\pm\varepsilon_i} \pm h_{ij}\phi(g_{\pm\varepsilon_i}))$, which entails $\phi(g_{\pm\varepsilon_i}) = \sigma g_{\pm\varepsilon_i}$. Since $g_{\pm\varepsilon_1}, \ldots, g_{\pm\varepsilon_n}$ are generators for $S(n)$, we obtain $\phi(g) = \sigma g$, $g \in S(n)$. This does imply that ϕ is trivial. The lemma is proved.

LEMMA 4. The superalgebra $H(n)$ has no nontrivial δ-derivations.

Proof. By α^* we mean $\alpha^* \not\supseteq \pm\varepsilon_i$. The argument above entails

$$\phi(h_i) = \psi(h_i) + \sum_{\alpha\in\Delta_0} b^{h_i}_{\alpha}, b^{h_i}_{\alpha} \in \bigcup_{k=1}^{[n/2]} H(n)_{2k}.$$

Consequently, $0 = \phi(h_ih_j) = \delta(\phi(h_i)h_j + h_i\phi(h_j))$, which implies $b^{h_i}_{\alpha^*} = 0$. For $\alpha^* \supseteq \varepsilon_j$, we have $b^{h_i}_{-\varepsilon_i+\alpha^*} = -b^{h_j}_{-\varepsilon_i+\alpha^*}$ and $b^{h_i}_{\varepsilon_i+\alpha^*} = b^{h_j}_{\varepsilon_i+\alpha^*}$. For $\alpha^* \supseteq -\varepsilon_j$, it is true that $b^{h_j}_{\varepsilon_i+\alpha^*} = -b^{h_i}_{\varepsilon_i+\alpha^*}$ and $b^{h_j}_{-\varepsilon_i+\alpha^*} = b^{h_i}_{-\varepsilon_i+\alpha^*}$.

Assume that for α^*, either $\alpha^* \supseteq \varepsilon_j$ or $\alpha^* \supseteq -\varepsilon_j$. We claim that $b^{h_i}_{\pm\varepsilon_i+\alpha^*} = 0$. Obviously,

$$\phi(g_{\varepsilon_i}) = \delta(\phi(h_i)g_{\varepsilon_i} + h_i\phi(g_{\varepsilon_i})), 0 = \phi(h_jg_{\varepsilon_i}) = \delta(\phi(h_j)g_{\varepsilon_i} + h_j\phi(g_{\varepsilon_i})),$$

from which we conclude that $\delta b^{h_i}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i} = b^{g_{\varepsilon_i}}_{\alpha^*} = -b^{h_j}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i} = b^{h_i}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i}$ for $\alpha^* \supseteq \varepsilon_j$; i.e., $b^{h_i}_{-\varepsilon_i+\alpha^*} = 0$. For $\alpha^* \supseteq -\varepsilon_j$, we have $(1+\delta)b^{h_i}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i} = (1+\delta)b^{h_j}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i} = (1+\delta)b^{g_{\varepsilon_i}}_{\alpha^*} = \delta b^{h_i}_{-\varepsilon_i+\alpha^*}g_{\varepsilon_i}$, which yields $b^{h_i}_{-\varepsilon_i+\alpha^*} = 0$.

Also it is easy to see that for $\alpha^* \supseteq \varepsilon_j$, it is true that

$$\phi(g_{-\varepsilon_i}) = -\delta(\phi(h_i)g_{-\varepsilon_i} + h_i\phi(g_{-\varepsilon_i})),\ 0 = \phi(h_j g_{-\varepsilon_i}) = \delta(\phi(h_j)g_{-\varepsilon_i} + h_j\phi(g_{-\varepsilon_i})),$$

whence $\delta b^{h_i}_{\varepsilon_i+\alpha^*} g_{-\varepsilon_i} = -b^{g_{-\varepsilon_i}}_{\alpha^*} = b^{h_j}_{\varepsilon_i+\alpha^*} g_{-\varepsilon_i} = b^{h_i}_{\varepsilon_i+\alpha^*} g_{-\varepsilon_i}$; i.e., $b^{h_i}_{\varepsilon_i+\alpha^*} = 0$.

That $b^{h_i}_{\varepsilon_i+\alpha^*} = 0$ for $\alpha^* \supseteq -\varepsilon_j$ derives from the equality $b^{h_j}_{-\varepsilon_j+\gamma^*} = 0$ proved above, where $\gamma^* \not\supseteq \pm\varepsilon_j$, $\gamma^* = \alpha^* + \varepsilon_i + \varepsilon_j$, and $b^{h_j}_{-\varepsilon_j+\gamma^*} = b^{h_i}_{\alpha^*+\varepsilon_i}$.

If $\delta \neq \frac{1}{2}$, then for $\gamma = 1, 2$ we have

$$\phi(g_{(-1)^\gamma\varepsilon_i}) = (-1)^\gamma\phi(h_i g_{(-1)^\gamma\varepsilon_i}) = (-1)^\gamma\delta h_i\phi(g_{(-1)^\gamma\varepsilon_i}).$$

This implies $\phi(g_{(-1)^\gamma\varepsilon_i}) = \sum\limits_{\alpha^*\in\Delta_0\cup\{0\}} b^{g_{(-1)^\gamma\varepsilon_i}}_{-(-1)^\gamma\varepsilon_i+\alpha^*}$ and $\delta = -1$. Note that

$$0 = \phi(g^2_{(-1)^\gamma\varepsilon_i}) = -2\sum_{\alpha^*\in\Delta_0\cup\{0\}} b^{g_{(-1)^\gamma\varepsilon_i}}_{-(-1)^\gamma\varepsilon_i+\alpha^*} g_{(-1)^\gamma\varepsilon_i};$$

i.e., $\phi(g_{(-1)^\gamma\varepsilon_i}) = 0$. Hence $\phi(g_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_t}-\varepsilon_{j_1}-\ldots-\varepsilon_{j_s}}) = 0$, whence $\phi(H(n)) = 0$.

If $\delta = \frac{1}{2}$ then $\phi(h_i) = \sigma h_i$. Consequently,

$$2\phi(g_{(-1)^\gamma\varepsilon_i}) = (-1)^\gamma 2\phi(h_i g_{(-1)^\gamma\varepsilon_i}) = \sigma g_{(-1)^\gamma\varepsilon_i} + (-1)^\gamma h_i\phi(g_{(-1)^\gamma\varepsilon_i});$$

i.e., $\phi(g_{(-1)^\gamma\varepsilon_i}) = \sigma g_{(-1)^\gamma\varepsilon_i}$. This does imply that $\phi(g_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_t}-\varepsilon_{j_1}-\ldots-\varepsilon_{j_s}}) = \sigma g_{\varepsilon_{i_1}+\ldots+\varepsilon_{i_t}-\varepsilon_{j_1}-\ldots-\varepsilon_{j_s}}$, which yields $\phi(g) = \sigma g$, $g \in H(n)$. Thus ϕ is trivial. The lemma is proved.

THEOREM 5. A Cartan-type Lie superalgebra does not have nontrivial δ-derivations.

The **proof** follows from Lemmas 2-4.

COROLLARY 6. A simple finite-dimensional Lie superalgebra over an algebraically closed field of characteristic 0 does not have nontrivial δ-derivations.

The **proof** follows from Theorem 5 and [5].

3. δ-SUPERDERIVATIONS OF SIMPLE FINITE-DIMENSIONAL SUPERALGEBRAS

Let A be a superalgebra. By a *superspace* we mean a $\mathbb{Z}_2$-graded space. A homogeneous element ψ of an endomorphism superspace $A \to A$ is called a *superderivation* if

$$\psi(xy) = \psi(x)y + (-1)^{p(x)\deg(\psi)}x\psi(y).$$

Suppose $\delta \in F$. A linear mapping $\phi : A \to A$ is called a δ-*superderivation* if

$$\phi(xy) = \delta(\phi(x)y + (-1)^{p(x)\deg(\phi)}x\phi(y)).$$

Consider a Lie superalgebra $G = G_{\bar{0}} + G_{\bar{1}}$ and fix an element $x \in G_{\bar{1}}$. Then $ad_x : y \to xy$ is a superderivation of G having parity $p(ad_x) = 1$. Obviously, for any superalgebra, multiplication by an element of the base field F is an even $\frac{1}{2}$-superderivation.

By the *supercentroid* $\Gamma_s(A)$ of a superalgebra A we mean a set of all homogeneous linear mappings $\chi : A \to A$ satisfying the condition

$$\chi(ab) = \chi(a)b = (-1)^{p(a)p(\chi)}a\chi(b)$$

for two arbitrary homogeneous elements $a, b \in A$.

A definition for a 1-superderivation coincides with the usual definition of a superderivation. A 0-superderivation is an arbitrary endomorphism ϕ of A such that $\phi(A^2) = 0$. A *nontrivial δ-superderivation* is a nonzero δ-superderivation which is not a 1- or 0-derivation, nor an element of the centroid. We are interested in how nontrivial δ-superderivations act on simple finite-dimensional Lie superalgebras and on simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0.

THEOREM 7. A simple finite-dimensional Lie superalgebra G over an algebraically closed field of characteristic 0 does not have nontrivial δ-superderivations.

Proof. It follows from Corollary 6 that simple finite-dimensional Lie superalgebras over an algebraically closed field of characteristic 0 have no nontrivial even δ-superderivations. We argue to show that nontrivial odd δ-superderivations likewise are missing.

Let ϕ be a nontrivial odd δ-superderivation. A map ψ_x on the superalgebra G is defined thus:

$$\psi_x = [\phi, ad_x] = \phi ad_x + ad_x \phi.$$

It is easy to see that the map ψ_x is a δ-superderivation. Thus, using Corollary 6 and keeping in mind that every even δ-superderivation is a δ-derivation of G, we conclude that ψ_x is trivial; i.e., $\psi_x = 0$, for $\delta \neq \frac{1}{2}$, and $\psi_x(g) = \alpha_x g$ for $\delta = \frac{1}{2}$, where $\alpha_x \in F$ and $g \in G$. For $\delta = \frac{1}{2}$ and $g \in G_{\bar{1}}$, we have

$$\alpha_x g = \psi_x(g) = \phi(xg) + x\phi(g) = \frac{1}{2}\phi(x)g + \frac{1}{2}x\phi(g) = -\psi_g(x) = -\alpha_g x;$$

i.e., $\alpha_x = 0$. Hence $\phi(x)g + x\phi(g) = 0$ for arbitrary g.

Note that $\phi(g_0 x) = \frac{1}{2}(\phi(g_0)x + g_0\phi(x)) = 0$ for $g_0 \in G_{\bar{0}}$. The fact that $G_{\bar{1}} = [G_{\bar{0}}, G_{\bar{1}}]$ entails $\phi(G_{\bar{1}}) = 0$. Now, with $G_{\bar{0}} = [G_{\bar{1}}, G_{\bar{1}}]$ in mind, we obtain $\phi = 0$.

If $\delta \neq \frac{1}{2}$, then

$$0 = \psi_x(g) = \phi(xg) + x\phi(g).$$

Hence $2\delta\phi(x)x = \phi(x^2) = \phi(x)x$ for $x \in G_{\bar{1}}$; in other words, $0 = \phi(x)x = \phi(x^2)$. It follows that $\phi(G_{\bar{0}}) = 0$. It remains to note that if $x \in G_{\bar{1}}$ and $g_0 \in G_{\bar{0}}$ then $\phi(xg_0) = -x\phi(g_0) = 0$ and $G_{\bar{1}} = [G_{\bar{0}}, G_{\bar{1}}]$. Consequently, ϕ is trivial. The theorem is proved.

The remaining part of the paper is a logical continuation of [4]. Therefore, we use the terms and notation developed therein. It is worthwhile recollecting the following identities for a Jordan algebra:

$$(x^2y)x = x^2(yx), \quad xy = yx. \tag{1}$$

LEMMA 8. A simple finite-dimensional Jordan superalgebra A with a semisimple even part over an algebraically closed field of characteristic 0 does not have nontrivial odd δ-superderivations.

Proof. We know that simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0 which contain unity e and have a semisimple even part are exhausted by superalgebras of the forms $M_{m,n}^{(+)}$, $Q(n)^{(+)}$, $osp(n, m)$, $P(n)$, $J(V, f)$, K_{10}, and D_t. A superalgebra K_3 with a semisimple even part contains no unity.

Let ϕ be a nontrivial odd δ-superderivation. Since $\phi(e) = \phi(ee) = 2\delta\phi(e)$, we have two options: $\delta \neq \frac{1}{2}$ or $\delta = \frac{1}{2}$. In the former case $\phi(e) = 0$, which entails $\phi(x) = \phi(ex) = \delta(\phi(e)x +$

$e\phi(x)) = \delta\phi(x)$, i.e., $\phi(x) = 0$. In the latter case $\phi(x) = \phi(ex) = \frac{1}{2}(\phi(e)x + e\phi(x))$, whence $\phi(x) = \phi(e)x$.

For the superalgebras $M_{m,n}^{(+)}$, $Q(n)^{(+)}$, $osp(n,m)$, $P(n)$, $J(V,f)$, D_t, and K_{10}, we will consider a Peirce decomposition with respect to idempotents e_i. The superalgebra A is representable as $A = A_0^{e_i} \oplus A_{\frac{1}{2}}^{e_i} \oplus A_1^{e_i}$, where $A_j^{e_i} = \{x \in A \mid xe_i = jx\}$, with $A_{\frac{1}{2}}^{e_i} \subseteq A_{\bar{1}}$. It is clear that $R_{e_j}R_{e_i} = R_{e_i}R_{e_j}$, where R_x is a right multiplication operator. Thus

$$0 = 2\phi(e_ie_j) = (\phi(e)e_i)e_j + e_i(\phi(e)e_j) = 2(\phi(e)e_i)e_j.$$

Using this, in view of the fact that $\phi(e) \in A_{\bar{1}}$, we will represent $\phi(e)$ as $\phi(e) = \sum \beta_\gamma g_\gamma$, where g_γ are basis elements. Clearly, if $g_\gamma \in A_{\frac{1}{2}}^{e_i} \cap A_{\frac{1}{2}}^{e_j}$, then $\beta_\gamma = 0$. Consequently, $\phi(e) = 0$ for $M_{m,n}^{(+)}$, $osp(n,m)$, $J(V,f)$, D_t, and K_{10}, which implies that odd δ-superderivations on these superalgebras are trivial. For $Q(n)^{(+)}$ and $P(n)$, we have $\phi(e) = \sum\limits_{i=1}^{n} \beta_i(e_{i,n+i} + e_{n+i,i})$ and $\phi(e) = \sum\limits_{i=1}^{n} \beta_i e_{n+i,i}$, respectively.

For the superalgebra $Q(n)^{(+)}$,

$$\begin{aligned}\frac{1}{2}(\beta_i(e_{i,n+i} + e_{n+i,i}) - \beta_j(e_{j,n+j} + e_{n+j,j})) &= \phi(\Delta^{i,j} \circ \Delta^{j,i}) \\ &= \frac{1}{2}((\phi(e) \circ \Delta^{i,j}) \circ \Delta^{j,i} - \Delta^{i,j} \circ (\phi(e) \circ \Delta^{j,i})) = 0,\end{aligned}$$

where $\Delta^{i,j} = e_{i,n+j} + e_{n+j,i}$, which yields $\beta_i = 0$.

For the superalgebra $P(n)$,

$$\begin{aligned}0 =& \phi((e_{i,j} + e_{n+j,n+i}) \circ (e_{j,n+i} - e_{i,n+j})) \\ =& \frac{1}{2}((\phi(e) \circ (e_{i,j} + e_{n+j,n+i})) \circ (e_{j,n+i} - e_{i,n+j}) \\ &+ (e_{i,j} + e_{n+j,n+i}) \circ (\phi(e) \circ (e_{j,n+i} - e_{i,n+j}))) \\ =& \frac{1}{4}(\beta_i(e_{n+j,i} + e_{n+i,j}) \circ (e_{j,n+i} - e_{i,n+j}) \\ &+ (e_{i,j} + e_{n+j,n+i}) \circ (\beta_j e_{i,j} - \beta_i e_{j,i} + \beta_j e_{n+j,n+i} - \beta_i e_{n+i,n+j})) \\ =& -\frac{1}{8}\beta_i(e_{j,j} + e_{n+j,n+j});\end{aligned}$$

i.e., $\beta_i = 0$. Therefore, $\phi(e) = 0$ for $Q(n)^{(+)}$ and $P(n)$, which gives $\phi = 0$.

For the superalgebra K_3, $\phi(e) = \delta(\phi(e)e + e\phi(e)) = \delta\phi(e)$, whence $\phi(e) = 0$. It is easy to see that $\phi(z) = \alpha_z e$ and $\phi(w) = \alpha_w e$. Hence

$$0 = 2\phi(e) = 2\phi([z,w]) = 2\delta(\phi(z)w - z\phi(w)) = \delta(\alpha_z w - \alpha_w z),$$

which implies $\phi = 0$. The lemma is proved.

We recall the definition of a superalgebra $J(\Gamma_n)$. Let Γ be the Grassmann algebra with a set of (odd) anticommutative generators $e_1, e_2, \ldots, e_n, \ldots$. To define a new multiplication, called the *Grassmann bracket*, we use the operation

$$\frac{\partial}{\partial e_j}(e_{i_1}e_{i_2}\ldots e_{i_n}) = \begin{cases}(-1)^{k-1}e_{i_1}e_{i_2}\ldots e_{i_{k-1}}e_{i_{k+1}}\ldots e_{i_n}, & \text{where } j = i_k, \\ 0, & \text{where } j \neq i_l,\ l = 1,\ldots,n.\end{cases}$$

Grassmann multiplication for $f, g \in \Gamma_0 \bigcup \Gamma_1$ is defined thus:

$$\{f, g\} = (-1)^{p(f)} \sum_{j=1}^{\infty} \frac{\partial f}{\partial e_j} \frac{\partial g}{\partial e_j}.$$

Let $\overline{\Gamma}$ be an isomorphic copy of Γ under an isomorphism mapping $x \to \overline{x}$. Consider a direct sum of vector spaces $J(\Gamma) = \Gamma + \overline{\Gamma}$, on which the structure of a Jordan superalgebra is defined by setting $A_0 = \Gamma_0 + \overline{\Gamma_1}$ and $A_1 = \Gamma_1 + \overline{\Gamma_0}$, with multiplication

$$a \bullet b = ab, \quad \overline{a} \bullet b = (-1)^{p(b)} \overline{ab}, \quad a \bullet \overline{b} = \overline{ab}, \quad \overline{a} \bullet \overline{b} = (-1)^{p(b)} \{a, b\},$$

where $a, b \in \Gamma_0 \bigcup \Gamma_1$ and ab is a product in Γ. Let Γ_n be a subalgebra of Γ generated by $e_1, e_2, \ldots, e_n$. Denote by $J(\Gamma_n)$ the subsuperalgebra $\Gamma_n + \overline{\Gamma_n}$ of $J(\Gamma)$. If $n \geqslant 2$, then $J(\Gamma_n)$ is a simple Jordan superalgebra.

LEMMA 9. The superalgebra $J(\Gamma_n)$ has no nontrivial odd δ-derivations.

Proof. Let ϕ be a nontrivial odd δ-superderivation and $\phi(1) = \alpha\gamma + \beta\overline{\nu}$, where $\alpha, \beta \in F$, $\gamma \in \Gamma$, and $\overline{\nu} \in \overline{\Gamma}$. Clearly,

$$\phi(x) = \phi(1 \bullet x) = \delta(\phi(1) \bullet x + \phi(x)),$$

whence $\phi(x) = \frac{\delta}{1-\delta}\phi(1) \bullet x$. Hence $\phi(1) = 0$ for $\delta \neq \frac{1}{2}$, and so $\phi(J(\Gamma_n)) = 0$. Consider the case $\delta = \frac{1}{2}$ in greater detail. Obviously,

$$\phi(1) = \phi(\overline{e_i} \bullet \overline{e_i}) = \frac{1}{2}(\phi(\overline{e_i}) \bullet \overline{e_i} + \overline{e_i} \bullet \phi(\overline{e_i})) = \phi(\overline{e_i}) \bullet \overline{e_i} = \overline{e_i} \bullet (\overline{e_i} \bullet \phi(1)).$$

For arbitrary x of the form $e_{i_1} e_{i_2} \ldots e_{i_k}$, we have

$$\overline{e_i} \bullet (\overline{e_i} \bullet x) = \begin{cases} x & \text{if } \frac{\partial x}{\partial e_i} = 0, \\ 0 & \text{if } \frac{\partial x}{\partial e_i} \neq 0, \end{cases} \tag{2}$$

$$\overline{e_i} \bullet (\overline{e_i} \bullet \overline{x}) = \begin{cases} \overline{x} & \text{if } \frac{\partial x}{\partial e_i} \neq 0, \\ 0 & \text{if } \frac{\partial x}{\partial e_i} = 0. \end{cases} \tag{3}$$

Let $\gamma = \gamma^{i+} + e_i \gamma^{i-}$ and $\overline{\nu} = \overline{\nu^{i+}} + e_i \overline{\nu^{i-}}$, where γ^{i-}, γ^{i+}, ν^{i-}, and ν^{i+} do not contain e_i. Since i is arbitrary, in view of (2) and (3), we have $\gamma = 1$ and $\nu = e_1 \ldots e_n$. Thus $\phi(1) = 0$ for $n = 2k$, whence $\phi(J(\Gamma_n)) = 0$ as well. If $n = 2k + 1$ then $\phi(1) = \beta\, \overline{e_1 \ldots e_n}$. Furthermore,

$$\begin{aligned} \phi(e_1) &= \phi(1) \bullet e_1 = \beta \overline{e_1 \ldots e_n} \bullet e_1 = 0, \\ \phi(\overline{e_1}) &= \phi(1) \bullet \overline{e_1} = \beta \overline{e_1 \ldots e_n} \bullet \overline{e_1} = -\beta e_2 \ldots e_n. \end{aligned}$$

The relations derived above yield $0 = \phi(e_1 \bullet \overline{e_1}) = \frac{1}{2}(e_1 \bullet \phi(\overline{e_1}) + \phi(e_1) \bullet \overline{e_1}) = -\frac{\beta}{2} e_1 \ldots e_n$; i.e., $\phi(1) = 0$. Consequently, $\phi(J(\Gamma_n)) = 0$. The lemma is proved.

THEOREM 10. A simple finite-dimensional Jordan superalgebra A over an algebraically closed field of characteristic 0 has no nontrivial δ-superderivations.

Proof. According to [8, 9], every simple finite-dimensional nontrivial Jordan superalgebra A over an algebraically closed field F of characteristic 0 is isomorphic to one of the following superalgebras: $M_{m,n}^{(+)}$, $Q(n)^{(+)}$, $osp(n, m)$, $P(n)$, $J(V, f)$, D_t, K_3, K_{10}, or $J(\Gamma_n)$. Even δ-superderivations are grading-preserving δ-derivations. It follows from [4] that nontrivial even

δ-superderivations are missing on this class of superalgebras. Lemmas 8 and 9 point to there being no nontrivial odd δ-superderivations for simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic 0. The theorem is proved

Let $A = A_0 \oplus A_1$ be a superalgebra over a field of characteristic distinct from 2, with multiplication $*$. On the vector space A, a new multiplication $\circ$ is given by the rule $a \circ b = \frac{1}{2}(a * b + (-1)^{p(a)p(b)} b * a)$, where $p(x)$ is the parity of an element x. Denote the resulting superalgebra by $A^{(+)}$.

COROLLARY 11. Let ϕ be a δ-superderivation of a superalgebra A over an algebraically closed field of characteristic 0, with $A^{(+)}$ a simple finite-dimensional Jordan superalgebra. Then ϕ is trivial.

Proof. The statement is a consequence of the fact that ϕ is a δ-superderivation of the superalgebra $A^{(+)}$. Note that

$$\begin{aligned} 2\phi(x \circ y) &= \phi(xy) + (-1)^{p(x)p(y)}\phi(yx) \\ &= \delta(\phi(x)y + (-1)^{p(y)p(\phi)}(-1)^{p(x)p(y)}y\phi(x) \\ &\quad +(-1)^{p(x)p(\phi)}x\phi(y) + (-1)^{p(x)p(y)}\phi(y)x) \\ &= 2\delta(\phi(x) \circ y + (-1)^{p(x)p(\phi)}x \circ \phi(y)). \end{aligned}$$

The result now follows by treating ϕ as a δ-superderivation of $A^{(+)}$ and using Theorem 10.

Noncommutative Jordan superalgebras are a natural generalization of the class of Jordan superalgebras. According to [10], the superalgebras satisfying the hypothesis of Corollary 11 are exemplified by simple finite-dimensional noncommutative Jordan superalgebras A of degree $t > 1$, where by a degree is meant a maximal number of nonzero pairwise orthogonal idempotents. Therefore, we have

THEOREM 12. A simple finite-dimensional noncommutative Jordan superalgebra A of degree $t > 1$ over an algebraically closed field of characteristic 0 does not have nontrivial δ-superderivations.

4. δ-DERIVATIONS OF SIMPLE FINITE-DIMENSIONAL JORDAN ALGEBRAS

δ-Derivations of semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic $\neq 2$ were described in [4]. We will give a description of δ-derivations for semisimple finite-dimensional algebras over a field of characteristic distinct from 2, and look at how the δ-derivations act on certain noncommutative Jordan algebras.

THEOREM 13. A semisimple finite-dimensional Jordan algebra A over a field of characteristic other than 2 has no nontrivial δ-derivations.

Proof. First we consider a partial case where A is a simple finite-dimensional Jordan algebra with unity 1. Let ϕ be a nontrivial δ-derivation of A. According to [4, Thm. 2.1], we have $\delta = \frac{1}{2}$ and $\phi(x) = \phi(1)x$. Let $Z(A)$ be the center of A. It is well known that if $Z(A)$ is a field then $\overline{Z(A)}$ is an algebraic closure of $Z(A)$ (see [11]). Consider $\overline{A} = A \otimes_{Z(A)} \overline{Z(A)}$. We know from [12] that $\overline{A}$ is a simple finite-dimensional Jordan algebra over an algebraically closed field $\overline{Z(A)}$.

We define a mapping $\overline{\phi} : \overline{A} \to \overline{A}$ by setting $\overline{\phi}(\sum x_i \otimes \alpha_i) = \sum \phi(1)x_i \otimes \alpha_i$, and show that $\overline{\phi}$ is a δ-derivation of $\overline{A}$.

Note that

$$\begin{aligned} \overline{\phi}((x \otimes \alpha)(y \otimes \beta)) &= \overline{\phi}(xy \otimes \alpha\beta) = \phi(xy) \otimes \alpha\beta \\ &= \tfrac{1}{2}(\phi(x)y \otimes \alpha\beta + x\phi(y) \otimes \alpha\beta) \\ &= \tfrac{1}{2}(\overline{\phi}(x \otimes \alpha)(y \otimes \beta) + (x \otimes \alpha)\overline{\phi}(y \otimes \beta)). \end{aligned}$$

That $\overline{\phi}$ is linear follows from the definition of $\overline{\phi}$.

Thus, in view of [4, Thm. 2.5], we obtain $\overline{\phi}(x) = \alpha x$, where $\alpha \in \overline{Z(A)}$. Hence $\phi(x) \otimes \beta = \overline{\phi}(x \otimes \beta) = \alpha x \otimes \beta$. From this, with $\phi : A \to A$ in mind, we derive $\phi(x) = \alpha x$, where $\alpha \in Z(A)$. Consequently, ϕ is trivial.

An argument for the general case repeats the proof for a semisimple Jordan algebra over an algebraically closed field of characteristic other than 2, presented in [4, Thm. 2.6]. The theorem is proved.

The algebras satisfying the identity $(x^2y)x = x^2(yx)$ are a natural generalization of the class of Jordan algebras. Whenever an algebra has unity, this identity readily transforms into a flexibility identity $(xy)x = x(yx)$. The algebras satisfying the two identities are said to be *noncommutative Jordan*. By a *degree* of a noncommutative Jordan algebra we mean a maximal number of nonzero pairwise orthogonal idempotents.

THEOREM 14. A simple finite-dimensional noncommutative Jordan algebra A of degree $t > 1$ over a field of characteristic distinct from 2 has no nontrivial δ-derivations.

Proof. Let ϕ be a nontrivial δ-derivation of A. If we appeal to the proof of Corollary 11 we see that ϕ is a δ-derivation of $A^{(+)}$. By [13], $A^{(+)}$ is a simple finite-dimensional Jordan algebra. Theorem 13 implies that for ϕ (treated as a δ-derivation of $A^{(+)}$), we have $\phi(x) = a \circ x$, where $a \in Z(A^{(+)})$. In view of [14], $Z(A) = Z(A^{(+)})$. Therefore, ϕ being a δ-derivation of A is trivial. The theorem is proved.

Let $\Delta_\delta(A)$ be a set of nontrivial δ-derivations of an algebra A and $\Gamma(A)$ be the centroid of A. For A, $A^{(-)}$ denotes an adjoint commutator algebra, i.e., one with multiplication $[a, b] = ab - ba$.

LEMMA 15. Let A be a unital flexible algebra over a field of characteristic distinct from 2. Then $\Delta_\delta(A) \subseteq \Gamma(A^{(-)})$.

Proof. According to [4, Thm. 2.1], nontrivial δ-derivations are possible only if $\delta = \frac{1}{2}$. Let $\phi \in \Delta_{\frac{1}{2}}(A)$ and e be unity in A. It is easy to see that

$$\phi((xy)x) = \frac{1}{2}(\phi(xy)x + (xy)\phi(x)) = \frac{1}{4}(\phi(x)y)x + \frac{1}{4}(x\phi(y))x + \frac{1}{2}(xy)\phi(x);$$

on the other hand,

$$\phi(x(yx)) = \frac{1}{2}(\phi(x)(yx) + x\phi(yx)) = \frac{1}{2}\phi(x)(yx) + \frac{1}{4}x(\phi(y)x) + \frac{1}{4}x(y\phi(x)).$$

In view of the flexibility identity, we have

$$2((xy)\phi(x) - \phi(x)(yx)) = x(y\phi(x)) - (\phi(x)y)x,$$

whence

$$\phi(x)x = x\phi(x)$$

for $y = e$. By linearizing the equality above, we obtain

$$[\phi(x), y] = [x, \phi(y)].$$

It remains to observe that

$$\begin{aligned}\phi([x, y]) = \phi(xy - yx) &= \frac{1}{2}(\phi(x)y - y\phi(x) + x\phi(y) - \phi(y)x) \\ &= \frac{1}{2}([\phi(x), y] + [x, \phi(y)]) = [\phi(x), y] = [x, \phi(y)].\end{aligned}$$

Hence $\Delta_{\frac{1}{2}}(A) \subseteq \Gamma(A^{(-)})$. The lemma is proved.

LEMMA 16. For any algebra A over a field of characteristic distinct from 2, $\Delta_\delta(A) = \Delta_\delta(A^{(+)}) \cap \Delta_\delta(A^{(-)})$ and $\Gamma(A) = \Gamma(A^{(+)}) \cap \Gamma(A^{(-)})$.

Proof. Obviously, $\Delta_\delta(A) \subseteq \Delta_\delta(A^{(+)}) \cap \Delta_\delta(A^{(-)})$ (which follows from an argument similar to one in Cor. 11). Let $\psi \in \Delta_\delta(A^{(+)}) \cap \Delta_\delta(A^{(-)})$; then

$$\psi(xy + yx) = \delta(\psi(x)y + y\psi(x) + x\psi(y) + \psi(y)x),$$
$$\psi(xy - yx) = \delta(\psi(x)y - y\psi(x) + x\psi(y) - \psi(y)x).$$

By summing these equalities, we arrive at

$$\psi(xy) = \delta(\psi(x)y + x\psi(y)).$$

This yields $\psi \in \Delta_\delta(A)$, which entails $\Delta_\delta(A) = \Delta_\delta(A^{(+)}) \cap \Delta_\delta(A^{(-)})$.

In a similar way, we derive $\Gamma(A) = \Gamma(A^{(+)}) \cap \Gamma(A^{(-)})$. The lemma is proved.

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Список литературы

[1] V. T. Filippov, "δ-Derivations of Lie algebras," *Sib. Mat. Zh.*, **39**, No. 6, 1409-1422 (1998).

[2] V. T. Filippov, "δ-Derivations of prime Lie algebras," *Sib. Mat. Zh.*, **40**, No. 1, 201-213 (1999).

[3] V. T. Filippov, "δ-Derivations of prime alternative and Mal'tsev algebras," *Algebra Logika*, **39**, No. 5, 618-625 (2000).

[4] I. B. Kaigorodov, "δ-Derivations of simple finite-dimensional Jordan superalgebras," *Algebra Logika*, **46**, No. 5, 585-605 (2007).

[5] I. B. Kaygorodov, "δ-Derivations of classical Lie superalgebras," *Sib. Mat. Zh.*, **50**, No. 3, 547-565 (2009).

[6] V. G. Kac, "Lie superalgebras," *Adv. Math.*, **26**, No. 1, 8-96 (1977).

[7] I. B. Penkov, "Characters of strongly generic irreducible Lie superalgebra representations," *Int. J. Math.*, **9**, No. 3, 331-366 (1998).

[8] I. L. Kantor, "Jordan and Lie superalgebras defined by the Poisson algebra," in *Algebra and Analysis* [in Russian], Tomsk State Univ., Tomsk (1989), pp. 55-80.

[9] V. G. Kac, "Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras," *Comm. Alg.*, **5**, 1375-1400 (1977).

[10] A. P. Pozhidaev and I. P. Shestakov, "Noncommutative Jordan superalgebras of degree $n \geq 2$," *Algebra Logika*, **49**, No. 1, 26-59 (2010).

[11] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings That Are Nearly Associative* [in Russian], Nauka, Moscow (1978).

[12] I. N. Herstein, *Noncommutative Rings, The Carus Math. Monogr.*, **15**, Math. Ass. Am. (1968).

[13] R. H. Oehmke, "On flexible algebras," *Ann. Math. (2)*, **68**, 221-230 (1958).

[14] V. G. Skosyrskii, "Strongly prime noncommutative Jordan algebras," *Tr. Inst. Mat. SO RAN*, **16**, 131-163 (1989).

[15] P. Zusmanovich, *On δ-derivations of Lie algebras and superalgebras*, arXiv:0907.2034v2.